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Research Article

Uniform Asymptotic Stability and Robust Stability for Positive Linear Volterra Difference Equations in Banach Lattices

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For positive linear Volterra difference equations in Banach lattices, the uniform asymptotic stability of the zero solution is studied in connection with the summability of the fundamental solution and the invertibility of the characteristic operator associated with the equations. Moreover, the robust stability is discussed and some stability radii are given explicitly.

1. Introduction

A dynamical system is called *positive* if any solution of the system starting from nonnegative states maintains nonnegative states forever. In many applications where variables represent nonnegative quantities we often encounter positive dynamical systems as mathematical models (see [1, 2]), and many researches for positive systems have been done actively; for recent developments see, for example, [3] and the references therein.

In this paper we treat the Volterra difference equations

$$x(n+1) = \sum_{j=0}^{n} Q(n-j)x(j), \tag{1.1}$$

together with

$$y(n+1) = \sum_{j=-\infty}^{n} Q(n-j)y(j) \tag{1.2}$$

in a (complex) Banach lattice X, where $\{Q(n)\}_{n\geq 0}$ is a sequence of compact linear operators on X satisfying the summability condition $\sum_{n=0}^{\infty}\|Q(n)\| < \infty$, and we study stability properties of (1.1) and (1.2) under the restriction that the operators $Q(n)$, $n \geq 0$, are positive. In fact, the restriction on $Q(n)$ yields the positivity for the above equations (whose notion is introduced in Section 2). Also, without the restriction, in [4] the authors characterized the uniform asymptotic stability of the zero solution of (1.1), together with (1.2), in connection with the invertibility of the characteristic operator

$$zI - \sum_{n=0}^{\infty} Q(n)z^{-n} \quad (I;\ \text{the identity operator on } X) \tag{1.3}$$

of (1.1) for any complex numbers z such that $|z| \geq 1$. In Section 3, we will prove that under the restriction that the operators $Q(n)$, $n \geq 0$, are positive, the invertibility of the characteristic operator reduces to that of the operator $zI - \sum_{n=0}^{\infty} Q(n)$, and consequently the uniform asymptotic stability of the zero solution for positive equations is equivalent to the condition which is much easier than the one for the characteristic operator in checking (Theorem 3.6). Moreover, we will discuss in Section 4 the robust stability of (1.1) and give explicit formulae of some stability radii.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{Z}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of natural numbers, nonnegative integers, nonpositive integers, integers, nonnegative real numbers, real numbers and complex numbers, respectively.

To make the presentation self-contained, we give some basic facts on Banach lattices which will be used in the sequel (see, e.g., [5]). Let $X_{\mathbb{R}} \neq \{0\}$ be a *real* vector space endowed with an order relation $\leq$. Then $X_{\mathbb{R}}$ is called an *ordered vector space*. Denote the *positive* elements of $X_{\mathbb{R}}$ by $X_+ := \{x \in X_{\mathbb{R}} : 0 \leq x\}$. If furthermore the *lattice property* holds, that is, if $x \vee y := \sup\{x, y\} \in X_{\mathbb{R}}$, for $x, y \in X_{\mathbb{R}}$, then $X_{\mathbb{R}}$ is called a *vector lattice*. It is important to note that X_+ is *generating*, that is,

$$X_{\mathbb{R}} = X_+ - X_+. \tag{2.1}$$

Then, the modulus of $x \in X_{\mathbb{R}}$ is defined by $|x| := x \vee (-x)$. If $\|\cdot\|$ is a norm on the vector lattice $X_{\mathbb{R}}$ satisfying the *lattice norm property*, that is, if

$$|x| \leq |y| \Longrightarrow \|x\| \leq \|y\|, \quad x, y \in X_{\mathbb{R}}, \tag{2.2}$$

then $X_{\mathbb{R}}$ is called a *normed vector lattice.* If, in addition, $(X_{\mathbb{R}}, \|\cdot\|)$ is a Banach space then $X_{\mathbb{R}}$ is called a (real) Banach lattice.

We now extend the notion of Banach lattices to the complex case. For this extension all underlying vector lattices $X_{\mathbb{R}}$ are assumed to be *relatively uniformly complete,* that is, if for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty}|\lambda_n| < +\infty$ and for every $x \in X_{\mathbb{R}}$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in $X_{\mathbb{R}}$ it holds that,

$$0 \le x_n \le \lambda_n x \Longrightarrow \sup_{n\in\mathbb{N}}\left(\sum_{j=1}^{n} x_j\right) \in X_{\mathbb{R}}. \tag{2.3}$$

Now, let $X_{\mathbb{R}}$ be a relatively uniformly complete vector lattice. The *complexification* of $X_{\mathbb{R}}$ is defined by $X = X_{\mathbb{R}} + iX_{\mathbb{R}}$. The modulus of $z = x + iy \in X$ is defined by

$$|z| = \sup_{0\le\theta\le 2\pi} |(\cos\theta)x + (\sin\theta)y| \in X_{\mathbb{R}}. \tag{2.4}$$

A complex vector lattice is defined as the complexification of a relatively uniformly complete vector lattice equipped with the modulus (2.4). If $X_{\mathbb{R}}$ is normed, then

$$\|x\| := \||x|\|, \quad x \in X \tag{2.5}$$

defines a norm on X satisfying the lattice norm property; in fact, the norm restricted to $X_{\mathbb{R}}$ is equivalent to the original norm in $X_{\mathbb{R}}$, and we use the same symbol $\|\cdot\|$ to denote the (new) norm. If $X_{\mathbb{R}}$ is a Banach lattice, then X equipped with the modulus (2.4) and the norm (2.5) is called a complex Banach lattice, and $X_{\mathbb{R}}$ is called the real part of X.

For Banach spaces E and F, we denote by $\mathcal{L}(E,F)$ the Banach space of all bounded linear operators from E to F equipped with the operator norm, and use the notation $\mathcal{L}(E)$ in place of $\mathcal{L}(E,E)$. Let E and F be Banach lattices with real parts $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$, respectively. An operator $T \in \mathcal{L}(E,F)$ is called *real* if $T(E_{\mathbb{R}}) \subset F_{\mathbb{R}}$. A linear operator T from E to F is called *positive*, denoted by $T \ge 0$, if $T(E_+) \subset F_+$ holds. Such an operator is necessarily bounded (see [5]) and hence real. Denote by $\mathcal{L}_{\mathbb{R}}(E,F)$ and $\mathcal{L}_+(E,F)$ the sets of real operators and positive operators between E and F, respectively:

$$\begin{aligned}\mathcal{L}_{\mathbb{R}}(E,F) &:= \{T \in \mathcal{L}(E,F) : T(E_{\mathbb{R}}) \subset F_{\mathbb{R}}\},\\ \mathcal{L}_+(E,F) &:= \{T \in \mathcal{L}(E,F) : T \ge 0\}.\end{aligned} \tag{2.6}$$

Then we observe that

$$|Tz| \le T|z| \quad \text{for } T \in \mathcal{L}_+(E,F),\ z \in E. \tag{2.7}$$

Indeed, it is clear that the inequality holds true for any $z \in E_{\mathbb{R}}$, the real part of E. Let $z = x + iy \in E$ with $x \in E_{\mathbb{R}}$ and $y \in E_{\mathbb{R}}$. Since $|z| \ge (\cos\theta)x + (\sin\theta)y \ge -|z|$ by (2.4), we get

$$T|z| \ge (\cos\theta)Tx + (\sin\theta)Ty \ge -T|z|, \quad 0 \le \theta \le 2\pi, \tag{2.8}$$

with $Tx \in E_{\mathbb{R}}$ and $Ty \in E_{\mathbb{R}}$. Then, it follows from the modulus (2.4) that

$$T|z| \geq \sup_{0\leq\theta\leq 2\pi} |(\cos\theta)Tx + (\sin\theta)Ty| = |Tx + iTy| = |Tz| \tag{2.9}$$

as required. We also emphasize the simple fact that

$$\|T\| = \sup_{x\in E_+,\ \|x\|=1} \|Tx\| \quad \text{for } T \in \mathcal{L}_+(E,F) \tag{2.10}$$

(see, e.g., [5, page 230]). Moreover, by the symbol $S \leq T$ we mean $T - S \geq 0$ for $T, S \in \mathcal{L}(E,F)$. Throughout this paper, X is assumed to be a complex Banach lattice with the real part $X_{\mathbb{R}}$ and the positive convex cone X_+.

For any interval $J \subset \mathbb{R}$, we use the same notation J meaning the discrete one $J \cap \mathbb{Z}$, for example, $[0,\sigma] = \{0,1,\ldots,\sigma\}$ for $\sigma \in \mathbb{Z}^+$. Also, for an X-valued function ξ on a discrete interval J, its norm is denoted by $\|\xi\|_J := \sup\{\|\xi(j)\| : j \in J\}$. Let $\sigma \in \mathbb{Z}^+$ and a function $\phi : [0,\sigma] \to X$ be given. We denote by $x(n;\sigma,\phi)$ the solution $x(n)$ of (1.1) satisfying $x(n) = \phi(n)$ on $[0,\sigma]$. Similarly, for $\tau \in \mathbb{Z}$ and a function $\psi : (-\infty,\tau] \to X$, we denote by $y(n;\tau,\psi)$ the solution $y(n)$ of (1.2) satisfying $y(n) = \psi(n)$ on $(-\infty,\tau]$. We then recall the representation formulae of solutions for initial value problems of (1.1) and (1.2) (see [6–8]). Let $\{R(n)\}$ be the fundamental solution of (1.1) (or (1.2)), that is, the sequence in $\mathcal{L}(X)$ satisfying

$$R(n+1) = \sum_{k=0}^{n-1} Q(n-k)R(k), \quad R(0) = I \tag{2.11}$$

for $n \in \mathbb{Z}^+$. Then, the solution $x(n;\sigma,\phi)$ is given by

$$x(n;\sigma,\phi) = R(n-\sigma)\phi(\sigma) + \sum_{k=\sigma}^{n-1} R(n-k-1)\left(\sum_{j=0}^{\sigma-1} Q(k-j)\phi(j)\right), \quad n \geq \sigma \tag{2.12}$$

for arbitrary initial function $\phi : [0,\sigma] \to X$, and also for arbitrary initial function $\psi : (-\infty,\tau] \to X$, the solution $y(n;\tau,\psi)$ of (1.2) is given by

$$y(n;\tau,\psi) = R(n-\tau)\psi(\tau) + \sum_{k=\tau}^{n-1} R(n-k-1)\left(\sum_{j=-\infty}^{\tau-1} Q(k-j)\psi(j)\right) \tag{2.13}$$

for $n \geq \tau$, where we promise $\sum_{k=m}^{m-1} := 0$ for $m \in \mathbb{Z}$.

Here, we give the definition of the positivity of Volterra difference equations.

Definition 2.1. Equation (1.1) is said to be *positive* if for any $\sigma \in \mathbb{Z}^+$ and $\phi : [0,\sigma] \to X_+$, the solution $x(n;\sigma,\phi) \in X_+$ for $n \geq \sigma$. Similarly, (1.2) is said to be positive if for any $\tau \in \mathbb{Z}$ and $\psi : (-\infty,\tau] \to X_+$, the solution $y(n;\tau,\psi) \in X_+$ for $n \geq \tau$.

Also, we follow the standard definitions for stabilities of the zero solution.

Definition 2.2. The zero solution of (1.1) is said to be

(i) *uniformly stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\sigma \in \mathbb{Z}^+$ and ϕ is an initial function on $[0,\sigma]$ with $\|\phi\|_{[0,\sigma]} < \delta$ then $\|x(n;\sigma,\phi)\| < \varepsilon$ for all $n \geq \sigma$;

(ii) *uniformly asymptotically stable* if it is uniformly stable, and if there exists a $\mu > 0$ such that, for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{Z}^+$ with the property that, if $\sigma \in \mathbb{Z}^+$ and ϕ is an initial function on $[0,\sigma]$ with $\|\phi\|_{[0,\sigma]} < \mu$ then $\|x(n;\sigma,\phi)\| < \varepsilon$ for all $n \geq \sigma + N$.

Definition 2.3. The zero solution of (1.2) is said to be

(i) *uniformly stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\tau \in \mathbb{Z}$ and ψ is an initial function on $(-\infty,\tau]$ with $\|\psi\|_{(-\infty,\tau]} < \delta$ then $\|y(n;\tau,\psi)\| < \varepsilon$ for all $n \geq \tau$;

(ii) *uniformly asymptotically stable* if it is uniformly stable, and if there exists a $\mu > 0$ such that, for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{Z}^+$ with the property that, if $\tau \in \mathbb{Z}$ and ψ is an initial function on $(-\infty,\tau]$ with $\|\psi\|_{(-\infty,\tau]} < \mu$ then $\|y(n;\tau,\psi)\| < \varepsilon$ for all $n \geq \tau + N$.

Here and subsequently, $\widetilde{Q}(z)$ denotes the Z-transform of $\{Q(n)\}$; that is, $\widetilde{Q}(z) := \sum_{n=0}^{\infty} Q(n)z^{-n}$, which is defined for $|z| \geq 1$ under our assumption $\sum_{n=0}^{\infty}\|Q(n)\| < \infty$. Then, $zI - \widetilde{Q}(z)$ is called the characteristic operator associated with (1.1) (or (1.2)). In [6, 7], under some restrictive conditions on $\{Q(n)\}$, we discussed the uniform asymptotic stability of the zero solution of (1.2) in connection with the summability of the fundamental solution $\{R(n)\}$, as well as the invertibility of the characteristic operator $zI - \widetilde{Q}(z)$; see also [9–12] for the case that X is finite dimensional. Moreover, we have shown in [4] the equivalence among these three properties without such restrictive conditions; more precisely, we have established the following.

Theorem 2.4 (see [4, Theorem 1]). *Let* $\{Q(n)\} \in l^1(\mathbb{Z}^+;\mathcal{L}(X))$, *and assume that* $Q(n)$, $n \in \mathbb{Z}^+$, *are all compact. Then the following statements are equivalent.*

(i) *The zero solution of* (1.1) *is uniformly asymptotically stable.*

(ii) *The zero solution of* (1.2) *is uniformly asymptotically stable.*

(iii) $\{R(n)\} \in l^1(\mathbb{Z}^+;\mathcal{L}(X))$.

(iv) *For any* z *such that* $|z| \geq 1$, *the operator* $zI - \widetilde{Q}(z)$ *is invertible in* $\mathcal{L}(X)$.

3. Stability for positive Volterra difference equations

In this section, we will prove that the uniform asymptotic stability of the zero solution of positive Volterra difference equations (1.1) and (1.2) is, in fact, characterized by the invertibility of the operator $zI - \widetilde{Q}(1)$ for $|z| \geq 1$. To this end we need some observations on the spectral radius of the Z-transform of the convolution kernel $\{Q(n)\}$.

First of all, we show the relation between the positivity of Volterra difference equations and that of the sequence of bounded linear operators $\{Q(n)\}$.

Proposition 3.1. *Equation* (1.1) *is positive if and only if all* $Q(n)$, $n \in \mathbb{Z}^+$, *are positive.*

Proof. Suppose that all $Q(n)$ are positive. Then from (2.11) each element $R(n)$ $(n \in \mathbb{Z}^+)$ of the fundamental solution is also positive; so that by virtue of (2.12), $x(n;\sigma,\phi) \in X_+$ for $n \geq \sigma$, where $\sigma \in \mathbb{Z}^+$ and $\phi : [0,\sigma] \to X_+$ are arbitrary. Hence, (1.1) is positive. Conversely, suppose (1.1) to be positive. Then we have in particular $x(1;0,\phi_0) = Q(0)\phi_0 \in X_+$ for $\phi_0 \in X_+$, which implies that $Q(0) \geq 0$. Let $\sigma > 0$ and for $k \in (0,\sigma]$, $\phi_k : [0,\sigma] \to X_+$ be any function such that $\phi_k(j) = 0$ except $j = \sigma - k$. Then it follows from (2.12) that

$$x(1+\sigma;\sigma,\phi_k) = \sum_{j=0}^{\sigma-1} Q(\sigma - j)\phi_k(j) = Q(k)\phi_k(\sigma - k) \in X_+, \tag{3.1}$$

which implies that $Q(k) \geq 0$ for $k \in (0,\sigma]$. Thus, $Q(n) \geq 0$ for $n \in \mathbb{Z}^+$ since $\sigma \in \mathbb{Z}^+$ is arbitrary. □

By using (2.13) one can verify the following proposition quite similarly.

Proposition 3.2. *Equation* (1.2) *is positive if and only if all* $Q(n)$, $n \in \mathbb{Z}^+$, *are positive.*

In what follows, we assume that $\{Q(n)\} \in l^1(\mathbb{Z}^+, \mathcal{L}(X))$ and each $Q(n)$ is compact. For any closed operator T on X we denote by $\sigma(T)$, $P_\sigma(T)$, and $\rho(T)$ the spectrum, the point spectrum, and the resolvent set of T, respectively. Also denote by $\operatorname{Int} D$ the interior of the unit disk D of the complex plane. Then for the uniform asymptotic stability of the zero solution of (1.1) we have the following criterion.

Theorem 3.3. *Suppose that* (1.1) *is positive. If* $\sigma(\widetilde{Q}(1)) \subset \operatorname{Int} D$, *the zero solution of* (1.1) *is uniformly asymptotically stable.*

Proof. In view of Theorem 2.4 it is sufficient to show that $zI - \widetilde{Q}(z)$ is invertible for $z \in \mathbb{C}$ with $|z| \geq 1$. Suppose by contradiction that $z_0 I - \widetilde{Q}(z_0)$ is not invertible for some z_0 with $|z_0| \geq 1$. Then $z_0 \in \sigma(\widetilde{Q}(z_0))$ and hence $z_0 \in P_\sigma(\widetilde{Q}(z_0))$ since $z_0 \neq 0$ and $\widetilde{Q}(z_0)$ is compact. Let x_0, $x_0 \neq 0$ be an eigenvector of the operator $\widetilde{Q}(z_0)$ for the eigenvalue z_0. Then, $\widetilde{Q}(z_0)x_0 = z_0 x_0$. We generally get $\widetilde{Q}(z_0)^k x_0 = z_0^k x_0$ for any $k \in \mathbb{N}$. Notice that the spectral radius of $\widetilde{Q}(1)$ is less than 1 by the assumption. Therefore, it follows from the well known Gelfand's formula (see, e.g., [13, Theorem 10.13]) for the spectral radius of bounded linear operators that $\lim_{k\to\infty}\|\widetilde{Q}(1)^k\|^{1/k} < 1$, which implies $\lim_{k\to\infty}\|\widetilde{Q}(1)^k\| = 0$. On the other hand, since $Q(n)$, $n \in \mathbb{Z}^+$, are positive by Proposition 3.1, we get

$$\left|\widetilde{Q}(z_0)x\right| = \left|\sum_{n=0}^{\infty} \frac{Q(n)}{z_0^n} x\right| \leq \sum_{n=0}^{\infty} \left|\frac{Q(n)}{z_0^n} x\right| \leq \sum_{n=0}^{\infty} Q(n)|x| = \widetilde{Q}(1)|x| \quad (\forall\, x \in X) \tag{3.2}$$

by (2.7); and generally,

$$\left|\widetilde{Q}(z_0)^k x\right| \leq \widetilde{Q}(1)\left|\widetilde{Q}(z_0)^{k-1} x\right| \leq \cdots \leq \widetilde{Q}(1)^k |x| \quad (\forall\, x \in X) \tag{3.3}$$

for any $k \in \mathbb{N}$. Therefore,

$$|x_0| \le |z_0^k x_0| = |\tilde{Q}(z_0)^k x_0| \le \tilde{Q}(1)^k |x_0|, \tag{3.4}$$

and it follows from the lattice norm property that

$$\|x_0\| \le \|\tilde{Q}(1)^k |x_0|\| \longrightarrow 0 \quad (k \longrightarrow \infty). \tag{3.5}$$

This is a contradiction, because we must get $x_0 = 0$ by (3.5). □

The converse of Theorem 3.3 also holds. To see this we need another proposition. Let $r(\lambda) := r(\tilde{Q}(\lambda))$ be the spectral radius of $\tilde{Q}(\lambda)$ for $\lambda \ge 1$.

Proposition 3.4. *Suppose that $Q(n)$, $n \in \mathbb{Z}^+$, are all positive. Then, $r(\lambda)$ is nonincreasing and continuous as a function on $[1, \infty)$.*

Proof. Let $\lambda_1 \ge \lambda_0 \ge 1$. Then,

$$\tilde{Q}(\lambda_1) = \sum_{n=0}^{\infty} \frac{Q(n)}{\lambda_1^n} \le \sum_{n=0}^{\infty} \frac{Q(n)}{\lambda_0^n} = \tilde{Q}(\lambda_0). \tag{3.6}$$

Observe that the resolvent $R(\lambda; \tilde{Q}(\lambda_i))$ of $\tilde{Q}(\lambda_i)$ $(i = 0, 1)$ is given by $R(\lambda; \tilde{Q}(\lambda_i)) = \sum_{n=0}^{\infty} \tilde{Q}(\lambda_i)^n / \lambda^{n+1}$ whenever $\lambda > \max\{r(\lambda_0), r(\lambda_1)\}$. Then, we deduce from (3.6) that

$$R(\lambda; \tilde{Q}(\lambda_1)) = \sum_{n=0}^{\infty} \frac{\tilde{Q}(\lambda_1)^n}{\lambda^{n+1}} \le \sum_{n=0}^{\infty} \frac{\tilde{Q}(\lambda_0)^n}{\lambda^{n+1}} = R(\lambda; \tilde{Q}(\lambda_0)). \tag{3.7}$$

Note also that under our assumption $\tilde{Q}(\lambda)$ is a positive operator for $\lambda \ge 1$ since X_+ is closed in X. In particular, $R(\lambda; \tilde{Q}(\lambda_1))$ is also positive and hence for $\lambda > \max\{r(\lambda_0), r(\lambda_1)\}$,

$$|R(\lambda; \tilde{Q}(\lambda_1))\xi| \le R(\lambda; \tilde{Q}(\lambda_1))|\xi| \le R(\lambda; \tilde{Q}(\lambda_0))|\xi|, \quad \xi \in X \tag{3.8}$$

by (2.7). Therefore $\|R(\lambda; \tilde{Q}(\lambda_1))\xi\| \le \|R(\lambda; \tilde{Q}(\lambda_0))|\xi|\| \le \|R(\lambda; \tilde{Q}(\lambda_0))\| \|\xi\|$ so that

$$\|R(\lambda; \tilde{Q}(\lambda_1))\| \le \|R(\lambda; \tilde{Q}(\lambda_0))\| \quad \text{for } \lambda > \max\{r(\lambda_0), r(\lambda_1)\}. \tag{3.9}$$

Now, let us assume that $r(\lambda_0) < r(\lambda_1)$ holds for some λ_0 and λ_1 with $\lambda_1 > \lambda_0 \ge 1$. Since $\tilde{Q}(\lambda_1)$ is positive, it follows from [5, Chapter 5, Proposition 4.1] that $r(\lambda_1) \in \sigma(\tilde{Q}(\lambda_1))$. Observe that if $\lambda \in \rho(\tilde{Q}(\lambda_1))$ and $|\mu - \lambda| \le 1/(2\|R(\lambda, \tilde{Q}(\lambda_1))\|)$, then $\mu I - \tilde{Q}(\lambda_1) = (I - (\lambda - \mu) R(\lambda, \tilde{Q}(\lambda_1)))(\lambda I - \tilde{Q}(\lambda_1))$ is invertible in $\mathcal{L}(X)$ with $R(\mu, \tilde{Q}(\lambda_1)) = R(\lambda, \tilde{Q}(\lambda_1)) \sum_{n=0}^{\infty} \{(\lambda - \mu) R(\lambda, \tilde{Q}(\lambda_1))\}^n$. Since $r(\lambda_1) \in \sigma(\tilde{Q}(\lambda_1))$, the above observation leads to the fact that $\|R(\lambda; \tilde{Q}(\lambda_1))\| \to \infty$ as $\lambda \to r(\lambda_1) + 0$; consequently, we get $\|R(\lambda; \tilde{Q}(\lambda_0))\| \to \infty$ as $\lambda \to r(\lambda_1) + 0$. On the other hand it follows from $\operatorname{dist}(\lambda, \sigma(\tilde{Q}(\lambda_0))) > 0$ for $\lambda \ge r(\lambda_1)$ that $R(\lambda; \tilde{Q}(\lambda_0)) \in \mathcal{L}(X)$ and the function $\lambda \mapsto$

$R(\lambda;\widetilde{Q}(\lambda_0)) \in \mathcal{L}(X)$ is continuous on $[r(\lambda_1),\infty)$. Hence, we get $\sup_{r(\lambda_1)\le\lambda\le r(\lambda_1)+1}\|R(\lambda;\widetilde{Q}(\lambda_0)\| < \infty$, which is a contradiction. Consequently, $r(\lambda_0) \ge r(\lambda_1)$ for $\lambda_0 \le \lambda_1$.

We next show the left continuity of $r(\lambda)$ on $(1,\infty)$. If $r(\lambda)$ is not left continuous at some $\lambda_0 > 1$, we have

$$r(\lambda_0) < \lim_{\varepsilon\to+0} r(\lambda_0-\varepsilon) =: r_-. \tag{3.10}$$

Since $r_- \in \rho(\widetilde{Q}(\lambda_0))$, we have $(r_-,\lambda_0) \in U := \{(r,\lambda) \in \mathbb{C}^2 : r \in \rho(\widetilde{Q}(\lambda)), |\lambda| > 1\}$. Notice that U is an open set. Hence, it follows that if $\varepsilon > 0$ is small enough, $(r(\lambda_0-\varepsilon),\lambda_0-\varepsilon) \in U$ that is, $r(\lambda_0-\varepsilon) \in \rho(\widetilde{Q}(\lambda_0-\varepsilon))$ for such an $\varepsilon > 0$. On the other hand, by virtue of [5, Chapter 5, Proposition 4.1] again, the positivity of $\widetilde{Q}(\lambda_0-\varepsilon)$ yields $r(\lambda_0-\varepsilon) \in \sigma(\widetilde{Q}(\lambda_0-\varepsilon))$; this is a contradiction.

$r(\lambda)$ is right continuous as well in $\lambda \in [1,\infty)$. Indeed, if it is not so, there exists a $\lambda_1 \ge 1$ such that

$$r_1 := r(\lambda_1) > \lim_{\varepsilon\to+0} r(\lambda_1+\varepsilon) =: r_+. \tag{3.11}$$

In view of the positivity of $\widetilde{Q}(\lambda_1)$ we have $r_1 \in \sigma(\widetilde{Q}(\lambda_1))$. Also, since $\widetilde{Q}(\lambda_1)$ is compact, r_1 is an isolated point of its spectrum $\sigma(Q(\lambda_1))$; in particular, there exists an $r_* \in (r_+,r_1)$ such that $[r_*,r_1) \subset \rho(\widetilde{Q}(\lambda_1))$. By the continuity of $\widetilde{Q}(z)$ there corresponds an $\varepsilon_0 > 0$ such that $r_*I - \widetilde{Q}(\lambda_1+\varepsilon)$ is invertible in $\mathcal{L}(X)$ for $0 \le \varepsilon \le \varepsilon_0$. Moreover, by the fact that $r_* > r(\lambda_1+\varepsilon)$ for $\varepsilon > 0$, one can see

$$R(r_*;\widetilde{Q}(\lambda_1+\varepsilon)) = \sum_{n=0}^{\infty}\frac{\widetilde{Q}(\lambda_1+\varepsilon)^n}{r_*^{n+1}}, \tag{3.12}$$

from which $R(r_*;\widetilde{Q}(\lambda_1+\varepsilon)) \ge 0$ readily follows because of the positivity of $\widetilde{Q}(\lambda_1+\varepsilon)$. Therefore, passing to the limit $\varepsilon \to +0$, we deduce that $R(r_*;\widetilde{Q}(\lambda_1)) \ge 0$. Let λ be any number such that $r_* < \lambda < r_1$. By the same reasoning, we see that $R(\lambda;\widetilde{Q}(\lambda_1)) \ge 0$. Since

$$R(\lambda;\widetilde{Q}(\lambda_1)) - R(r_*;\widetilde{Q}(\lambda_1)) = (r_*-\lambda)R(\lambda;\widetilde{Q}(\lambda_1))R(r_*;\widetilde{Q}(\lambda_1)), \tag{3.13}$$

we get

$$\begin{aligned} R(r_*;\widetilde{Q}(\lambda_1)) &= R(\lambda;\widetilde{Q}(\lambda_1)) + (\lambda-r_*)R(\lambda;\widetilde{Q}(\lambda_1))R(r_*;\widetilde{Q}(\lambda_1)) \\ &\ge R(\lambda;\widetilde{Q}(\lambda_1)). \end{aligned} \tag{3.14}$$

Then, for $x \in X$, it follows from (2.7) that

$$|R(\lambda;\widetilde{Q}(\lambda_1))x| \le R(\lambda;\widetilde{Q}(\lambda_1))|x| \le R(r_*;\widetilde{Q}(\lambda_1))|x|, \tag{3.15}$$

hence

$$\|R(\lambda;\widetilde{Q}(\lambda_1))\| \leq \|R(r_*;\widetilde{Q}(\lambda_1))\| \tag{3.16}$$

for any λ with $r_* < \lambda < r_1$. This is a contradiction, because $\|R(\lambda;\widetilde{Q}(\lambda_1))\| \to \infty$ as $\lambda \to r_1$. The proof is now completed. □

Theorem 3.5. *Suppose that* (1.1) *is positive. If the zero solution of* (1.1) *is uniformly asymptotically stable, then* $\sigma(\widetilde{Q}(1)) \subset \operatorname{Int} D$.

Proof. Set $f(\lambda) := \lambda - r(\lambda)$. To prove the theorem it is sufficient to show $f(1) > 0$. Assume that $f(1) \leq 0$. Since by Proposition 3.4 $f(\lambda)$ is continuous, there exists $\lambda_1 \geq 1$ such that $f(\lambda_1) = 0$, that is, $\lambda_1 = r(\widetilde{Q}(\lambda_1))$. It follows from the positivity of $\widetilde{Q}(\lambda_1)$, together with its compactness, that λ_1 belongs to $P_\sigma(\widetilde{Q}(\lambda_1))$; hence, there exists an $x_1 \neq 0$ such that $\widetilde{Q}(\lambda_1)x_1 = \lambda_1 x_1$, or equivalently

$$\sum_{k=0}^{\infty} \frac{Q(k)}{\lambda_1^k} x_1 = \lambda_1 x_1. \tag{3.17}$$

Setting $y(n) := \lambda_1^n x_1$, we have

$$y(n+1) = \lambda_1^{n+1} x_1 = \sum_{k=0}^{\infty} Q(k)\lambda_1^{n-k} x_1 = \sum_{k=0}^{\infty} Q(k) y(n-k) = \sum_{k=-\infty}^{n} Q(n-k) y(k), \tag{3.18}$$

so that $y(n)$ is a solution of (1.2). By virtue of Theorem 2.4 and our assumption, (1.2) is uniformly asymptotically stable and therefore $y(n) \to 0$ as $n \to \infty$, which is impossible because $\|y(n)\| \geq \|x_1\|$ for all $n \in \mathbb{Z}^+$. Thus we must have $f(1) > 0$. □

Combining the results above with Theorem 2.4, we have, for positive Volterra difference equations, the equivalence among the uniform asymptotic stability of the zero solution of (1.1) and (1.2), the summability of the fundamental solution and the invertibility of the operator $zI - \widetilde{Q}(1)$ outside the unit disk.

Theorem 3.6. *Let the assumptions in Theorem 2.4 hold. If, in addition,* $Q(n)$ *are all positive, then the following statements are equivalent.*

(i) *The zero solution of* (1.1) *is uniformly asymptotically stable.*

(ii) *The zero solution of* (1.2) *is uniformly asymptotically stable.*

(iii) $\{R(n)\} \in l^1(\mathbb{Z}^+;\mathcal{L}(X))$.

(iv) *The operator* $zI - \widetilde{Q}(1)$ *is invertible in* $\mathcal{L}(X)$ *for* $|z| \geq 1$.

Before concluding this section, we will give an example to which our Theorem 3.6 is applicable. In [6, 7], following the idea in [14], we have shown that Volterra difference equations on a Banach space X are naturally derived from abstract differential equations on X with piecewise continuous delays of type

$$\dot{u}(t) = Au(t) + \sum_{k=0}^{\infty} B(k)u([t-k]), \quad t \geq 0, \tag{3.19}$$

where $[\cdot]$ denotes the Gaussian symbol and A is the inifinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, of bounded linear operators on X, and $B(k)$, $k \in \mathbb{Z}^+$ are bounded linear operators on X such that

$$\sum_{k=0}^{\infty} \|B(k)\| < \infty. \tag{3.20}$$

Recall that a function $u : \mathbb{Z}^- \cup [0,\infty) \to X$ with $\sup_{\theta\in\mathbb{Z}^-} \|u(\theta)\| < \infty$ is called a (mild) solution of (3.19) on $[0,\infty)$, if u is continuous on $[0,\infty)$, and satisfies the relation

$$u(t) = T(t-\sigma)u(\sigma) + \int_{\sigma}^{t} T(t-s)\left(\sum_{k=0}^{\infty} B(k)u([s-k])\right)ds, \quad t \geq \sigma \geq 0. \tag{3.21}$$

In case of $n \leq t < n+1$ for some $n \in \mathbb{Z}^+$, the relation above yields that

$$\begin{aligned} u(t) &= T(t-n)u(n) + \int_{n}^{t} T(t-s)\left(\sum_{k=0}^{\infty} B(k)u([s-k])\right)ds \\ &= T(t-n)u(n) + \sum_{k=0}^{\infty}\left(\int_{n}^{t} T(t-s)B(k)u(n-k)ds\right). \end{aligned} \tag{3.22}$$

Letting $t \to n+1$, we get the Volterra difference equation

$$u(n+1) = \sum_{k=0}^{\infty} Q(k)u(n-k), \quad n \in \mathbb{Z}^+, \tag{3.23}$$

where $Q(k)$, $k \in \mathbb{Z}^+$, are bounded linear operators on X defined by

$$Q(0)x = T(1)x + \int_{0}^{1} T(\tau)B(0)x\,d\tau, \qquad Q(k)x = \int_{0}^{1} T(\tau)B(k)x\,d\tau, \quad k = 1,2,\ldots \tag{3.24}$$

for $x \in X$. Conversely, if u satisfies (3.23) with $\sup_{\theta\in\mathbb{Z}^-} \|u(\theta)\| < \infty$, then the function u extended to nonintegers t by the relation

$$u(t) = T(t-n)u(n) + \sum_{k=0}^{\infty}\left(\int_{n}^{t} T(t-s)B(k)u(n-k)ds\right), \quad n < t < n+1, \ n \in \mathbb{Z}^+ \tag{3.25}$$

is a (mild) solution of (3.19). Thus, abstract differential equations of type (3.19) lead to Volterra difference equations on X.

Now suppose that the semigroup $T(t)$ is compact. Then, $T(t)$ is continuous in $t > 0$ with respect to the operator norm ([15]) and also $Q(k)$, $k \in \mathbb{Z}^+$, defined by the relation (3.24) are compact operators on X (see [6, Proposition 1]). Moreover, it follows from (3.20) that $\{Q(k)\} \in l^1(\mathbb{Z}^+; \mathcal{L}(X))$. Moreover in the restricted case where $B(k)$, $k \in \mathbb{Z}^+$, are given by $B(k) \equiv b(k)I$, $k \in \mathbb{Z}^+$, for some $b(k) \in \mathbb{C}$ with $\sum_{k=0}^{\infty}|b(k)| < \infty$, we know by [7, Proposition 1] that the spectrum of the characteristic operator $zI - \widetilde{Q}(z)$ of (3.23) is given by the formula:

$$\sigma(zI - \widetilde{Q}(z)) = \{z\} \cup \left\{ z - e^{\nu} - \widetilde{b}(z)\int_0^1 e^{\nu\tau}d\tau : \nu \in \sigma(A) \right\} \quad \text{for } |z| \geq 1. \tag{3.26}$$

Hence, in the restricted case, Theorem 2.4 implies that the zero solution of (3.23) is uniformly asymptotically stable if and only if

$$z \neq e^{\nu} + \widetilde{b}(z)\int_0^1 e^{\nu\tau}d\tau \tag{3.27}$$

for all $\nu \in \sigma(A)$ and $|z| \geq 1$.

We further assume that X is a complex Banach lattice, the compact semigroup $T(t)$ on X is positive, and that $b(k)$, $k \in \mathbb{Z}^+$, are all nonnegative. Then the sequence $\{Q(k)\}$ defined by (3.24) meets the assumptions in Theorem 3.6. Noticing that $\sigma(\widetilde{Q}(1)) = \{0\} \cup \{e^{\nu} + \widetilde{b}(z)\int_0^1 e^{\nu\tau}d\tau : \nu \in \sigma(A)\}$, we know by Theorem 3.6 in the further restricted case that the zero solution of (3.23) is uniformly asymptotically stable if and only if

$$\left| e^{\nu} + \widetilde{b}(1)\int_0^1 e^{\nu\tau}d\tau \right| < 1 \tag{3.28}$$

for all $\nu \in \sigma(A)$.

4. Robust stability and some stability radii of positive Volterra difference equations

Let (1.1) be uniformly asymptotically stable, that is, the zero solution of (1.1) is uniformly asymptotically stable, and consider a perturbed difference equation of the form

$$x(n+1) = \sum_{j=0}^{n}(Q(n-j) + D\Gamma(n-j)E)x(j), \quad n \in \mathbb{Z}^+, \tag{4.1}$$

where $D \in \mathcal{L}(U, X)$, $E \in \mathcal{L}(X, V)$ are given operators corresponding to the structure of perturbations and $\Gamma = \{\Gamma(n)\} \in l^1(\mathbb{Z}^+, \mathcal{L}(V, U))$ is an unknown (disturbance) parameter. Here U and V are also assumed to be complex Banach lattices. Our objective in this section is to determine various stability radii of (1.1) provided that $Q(n)$ are all positive; for this topic in case that the space X is finite dimensional, see, for example, [16] and the references therein. By the stability radius of (1.1) we mean the supremum of positive numbers α such that the uniform asymptotic stability of the perturbed (4.1) persists whenever the size of the perturbation $\Gamma = \{\Gamma(n)\}$, measured by the l^1-norm $\|\Gamma\|_{l^1}$, is less than α (for precise definitions see the paragraph preceding Theorem 4.3).

Here and hereafter we also assume that for any perturbation Γ, $\Gamma(n)$, $n \in \mathbb{Z}^+$, are all compact, although this assumption is not necessary in the case that at least one of the operators D and E is compact. In what follows, we define $1/0 = +\infty$ by convention.

Theorem 4.1. *Let $Q(n)$, $n \in \mathbb{Z}^+$, be positive. Suppose that* (1.1) *is uniformly asymptotically stable and D, E are both positive. Then the perturbed* (4.1) *is still uniformly asymptotically stable if*

$$\|\Gamma\|_{l^1} < \frac{1}{\left\|E\left(I - \widetilde{Q}(1)\right)^{-1} D\right\|}. \tag{4.2}$$

We need the following lemma to prove the theorem.

Lemma 4.2. *Under the same assumptions as in Theorem 4.1 we have*

$$\sup_{|z|\geq 1} \left\|E\left(zI - \widetilde{Q}(z)\right)^{-1} D\right\| = \left\|E\left(I - \widetilde{Q}(1)\right)^{-1} D\right\|. \tag{4.3}$$

Proof. Since (1.1) is uniformly asymptotically stable, $r(\widetilde{Q}(1))$, the spectral radius of $\widetilde{Q}(1)$, is less than 1 by Theorem 3.5. In particular, it follows that $\sum_{n=0}^{\infty} \widetilde{Q}(1)^n$ is convergent and coincides with $(I - \widetilde{Q}(1))^{-1}$. Let $z \in \mathbb{C}$, $|z| \geq 1$, be given. As in the proof of Theorem 3.3, one can see $|\widetilde{Q}(z)^k x| \leq \widetilde{Q}(1)^k |x|$ for $k \in \mathbb{N}$ and $x \in X$; hence $\|\widetilde{Q}(z)^k\| \leq \|\widetilde{Q}(1)^k\|$, so that $r(\widetilde{Q}(z)) \leq r(\widetilde{Q}(1)) < 1$. Therefore,

$$\left|\left(zI - \widetilde{Q}(z)\right)^{-1} x\right| = \left|\sum_{n=0}^{\infty} \frac{\widetilde{Q}(z)^n}{z^{n+1}} x\right| \leq \sum_{n=0}^{\infty} \left|\frac{\widetilde{Q}(z)^n x}{z^{n+1}}\right| \leq \sum_{n=0}^{\infty} \widetilde{Q}(1)^n |x| = \left(I - \widetilde{Q}(1)\right)^{-1} |x|. \tag{4.4}$$

The positivity of D and E then implies

$$\left|E\left(zI - \widetilde{Q}(z)\right)^{-1} Du\right| \leq E\left|\left(zI - \widetilde{Q}(z)\right)^{-1} Du\right| \leq E\left(I - \widetilde{Q}(1)\right)^{-1} D|u| \tag{4.5}$$

for $u \in U$, and we thus obtain

$$\left\|E\left(zI - \widetilde{Q}(z)\right)^{-1} D\right\| \leq \left\|E\left(I - \widetilde{Q}(1)\right)^{-1} D\right\| \quad \text{for } |z| \geq 1, \tag{4.6}$$

which completes the proof. □

Proof of Theorem 4.1. Assume that perturbed (4.1) is not uniformly asymptotically stable for some $\Gamma \in l^1(\mathbb{Z}^+, \mathcal{L}(V, U))$ satisfying $\|\Gamma\|_{l^1} < 1/\|E(I - \widetilde{Q}(1))^{-1} D\|$, that is,

$$\|\Gamma\|_{l^1} \left\|E\left(I - \widetilde{Q}(1)\right)^{-1} D\right\| < 1. \tag{4.7}$$

Then, by Theorem 2.4 there exists a $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$ such that $z_0 I - \tilde{Q}(z_0) - D\tilde{\Gamma}(z_0)E$ is not invertible. So $z_0 \in \sigma(\tilde{Q}(z_0) + D\tilde{\Gamma}(z_0)E) = P_\sigma(\tilde{Q}(z_0) + D\tilde{\Gamma}(z_0)E)$ since $D\tilde{\Gamma}(z_0)E$ is compact. Hence, there corresponds an $x \in X$ with $x \neq 0$ satisfying $(z_0 I - \tilde{Q}(z_0))x = D\tilde{\Gamma}(z_0)Ex$. By virtue of the uniform asymptotic stability of (1.1) we know that $z_0 I - \tilde{Q}(z_0)$ is invertible; and therefore we get

$$\left(z_0 I - \tilde{Q}(z_0)\right)^{-1} D\tilde{\Gamma}(z_0)Ex = x, \tag{4.8}$$

so that $E(z_0 I - \tilde{Q}(z_0))^{-1} D\tilde{\Gamma}(z_0)Ex = Ex$ and $Ex \neq 0$. In view of Lemma 4.2,

$$\|Ex\| \leq \left\|E\left(z_0 I - \tilde{Q}(z_0)\right)^{-1} D\right\| \left\|\tilde{\Gamma}(z_0)\right\| \|Ex\| \leq \left\|E\left(I - \tilde{Q}(1)\right)^{-1} D\right\| \|\Gamma\|_{l^1} \|Ex\|, \tag{4.9}$$

which gives $\|\Gamma\|_{l^1} \|E(I - \tilde{Q}(1))^{-1} D\| \geq 1$, a contradiction to (4.7). The proof is completed. □

Let $\mathcal{K}(V, U) \subset \mathcal{L}(V, U)$ be the set of all compact operators mapping V into U. We introduce three classes of perturbations defined as $\mathcal{P}_{\mathbb{C}} = l^1(\mathbb{Z}^+, \mathcal{K}(V, U))$, $\mathcal{P}_{\mathbb{C}} = l^1(\mathbb{Z}^+, \mathcal{L}_{\mathbb{R}}(V, U) \cap \mathcal{K}(V, U))$ and $\mathcal{P}_+ = l^1(\mathbb{Z}^+, \mathcal{L}_+(V, U) \cap \mathcal{K}(V, U))$. Then the complex, real and positive stability radius of (1.1) under perturbations is defined, respectively, by

$$\begin{aligned} r_{\mathbb{C}} &= \inf\{\|\Gamma\|_{l^1} : \Gamma \in \mathcal{P}_{\mathbb{C}}, (4.1) \text{ is not uniformly asymptotically stable}\}, \\ r_{\mathbb{R}} &= \inf\{\|\Gamma\|_{l^1} : \Gamma \in \mathcal{P}_{\mathbb{R}}, (4.1) \text{ is not uniformly asymptotically stable}\}, \\ r_{+} &= \inf\{\|\Gamma\|_{l^1} : \Gamma \in \mathcal{P}_{+}, (4.1) \text{ is not uniformly asymptotically stable}\}, \end{aligned} \tag{4.10}$$

where the convention $\inf \emptyset = +\infty$ is used. By definition, it is easy to see that $r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_+$. On the other hand Theorem 4.1 yields the estimate

$$r_{\mathbb{C}} \geq \frac{1}{\left\|E\left(I - \tilde{Q}(1)\right)^{-1} D\right\|}, \tag{4.11}$$

provided that the assumptions of the theorem are satisfied. In fact, these three radii coincide, that is, we have the following.

Theorem 4.3. *Let $Q(n)$, $n \in \mathbb{Z}^+$, be positive. Suppose that* (1.1) *is uniformly asymptotically stable and D, E are both positive. Then*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\left\|E\left(I - \tilde{Q}(1)\right)^{-1} D\right\|}. \tag{4.12}$$

Proof. By the fact $r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_+$, combined with (4.11), it is sufficient to prove that

$$r_+ \leq \frac{1}{\left\|E\left(I - \tilde{Q}(1)\right)^{-1} D\right\|}. \tag{4.13}$$

We may consider the case $E(I-\widetilde{Q}(1))^{-1}D \neq 0$ since otherwise the theorem is trivial. Suppose by contradiction that (4.13) does not hold. Then there is an $\varepsilon > 0$, $\varepsilon < \|E(I-\widetilde{Q}(1))^{-1}D\|$, such that

$$r_+ > \frac{1}{\left\|E\left(I-\widetilde{Q}(1)\right)^{-1}D\right\| - \varepsilon}. \tag{4.14}$$

By the same reasoning as in the proof of Lemma 4.2, one can see that $(I-\widetilde{Q}(1))^{-1} = \sum_{n=0}^{\infty}\widetilde{Q}(1)^n \geq 0$ and hence $E(I-\widetilde{Q}(1))^{-1}D \in \mathcal{L}_+(U,V)$. Also by (2.10) one may choose $u_0 \in U_+$ with $\|u_0\| = 1$ such that

$$\left\|E\left(I-\widetilde{Q}(1)\right)^{-1}Du_0\right\| \geq \left\|E\left(I-\widetilde{Q}(1)\right)^{-1}D\right\| - \varepsilon. \tag{4.15}$$

Now let $v_0 := E(I-\widetilde{Q}(1))^{-1}Du_0$. Since $v_0 \in V_+$, there exists a positive $f \in V^*$, $\|f\| = 1$, satisfying $f(v_0) = \|v_0\| = \|E(I-\widetilde{Q}(1))^{-1}Du_0\|$ (see [17, Proposition 1.5.7]). Consider a map $\Phi : V \to U$ defined by

$$\Phi(v) := \frac{f(v)}{\left\|E\left(I-\widetilde{Q}(1)\right)^{-1}Du_0\right\|}u_0, \quad v \in V. \tag{4.16}$$

Then, it is easy to see that $\Phi \in \mathcal{L}_+(V,U) \cap \mathcal{K}(V,U)$ and $\|\Phi\| = 1/\|E(I-\widetilde{Q}(1))^{-1}Du_0\|$. Hence,

$$\|\Phi\| \leq \frac{1}{\left\|E\left(I-\widetilde{Q}(1)\right)^{-1}D\right\| - \varepsilon}. \tag{4.17}$$

By setting $x_0 := (I-\widetilde{Q}(1))^{-1}Du_0$, we have

$$\Phi Ex_0 = \Phi v_0 = \frac{f(v_0)}{\left\|E\left(I-\widetilde{Q}(1)\right)^{-1}Du_0\right\|}u_0 = \frac{\|v_0\|}{\left\|E\left(I-\widetilde{Q}(1)\right)^{-1}Du_0\right\|}u_0 = u_0. \tag{4.18}$$

Notice that $x_0 \neq 0$ because $u_0 \neq 0$, and that $x_0 = (I-\widetilde{Q}(1))^{-1}D\Phi Ex_0$, or equivalently $(I-\widetilde{Q}(1) - D\Phi E)x_0 = 0$. Define a perturbation $\Gamma_* \in l^1(\mathbb{Z}^+, \mathcal{K}(V,U))$ by

$$\Gamma_*(n) := 2^{-n-1}\Phi, \quad n \in \mathbb{Z}^+. \tag{4.19}$$

Then $\Gamma_* \in \mathcal{D}_+$. Moreover,

$$\left(I-\widetilde{Q}(1) - D\widetilde{\Gamma}_*(1)E\right)x_0 = \left(I-\widetilde{Q}(1) - D\Phi E\right)x_0 = 0, \tag{4.20}$$

and in particular $1 \in \sigma(\widetilde{Q}(1) + D\widetilde{\Gamma}_*(1)E)$, which means, by Theorem 3.6 (or Theorem 2.4), that the perturbed (4.1) with $\Gamma = \Gamma_*$ is not uniformly asymptotically stable. Therefore,

$$r_+ \leq \left\|\Gamma_*\right\|_{l^1} = \|\Phi\| \leq \frac{1}{\left\|E\left(I - \widetilde{Q}(1)\right)^{-1} D\right\| - \varepsilon}, \tag{4.21}$$

which contradicts (4.14). Consequently we must have (4.13), and this completes the proof. □

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